A topological vector space that is not metrizable.

Question: Does there exist a topological vector space that is not metrizable?

Yes.

A typical example is $C[0,1]$, where $[0,1]$ is equipped the usual topology, and the topology on $C[0,1]$ is given by "$f_n \to f$ if and only if $f_n(x) \to f(x)$ for all $x \in X$". This topology is also called the "weak topology".

On $C[0,1]$, use π to denote the "finest" (roughly speaking, with most open sets) topology that satisfies: if $f_n(x) \to f(x)$ in the sense of the normal topology on $\mathbb{R}$ for all $x \in X$, then $f_n \to f$ in the sense of π.

Then $(C[0,1], \pi)$ is a topological space that is not metrizable.